Chapter 3: Discrete Random Variable

Shiwen Shen

University of South Carolina

2017 Summer

Random Variable

- **Definition:** A random variable is a function from a sample space S into the real numbers. We usually denote random variables with uppercase letters, e.g. X, Y ...

$$X : S \to \mathbb{R}.$$

- Example:
  1. Experiment: toss a coin 2 times
     Random variable: $X =$ number of heads in 2 tosses
  2. Experiment: toss 10 dice
     Random variable: $X =$ sum of the numbers
  3. Experiment: apply different amounts of fertilizer to corn plants
     Random variable: $X =$ yield/acre
- Remark: probability is also a function mapping events in the sample space to real numbers. One reason to define random variable is that it is easier to calculate probabilities with random variable instead of events.

Sample Space of Random Variable

- In defining a random variable, we have also defined a new sample space. For example, in the experiment of tossing a coin 2 times, the original sample space is

$$S = \{HH, HT, TH, TT\}$$

- We define a random variable X = number of heads in 2 tosses. The new sample space is

$$\chi = \{0, 1, 2\}$$

- The new sample space χ is called the range of the random variable X.

Type of Random Variables

- A **discrete** random variable can take one of a countable list of distinct values. It's sample space has finite or countable outcomes.
- A **continuous** random variable can take any value in an interval of the real number line. It's sample space has uncountable outcomes.
- Classify the following random variables as discrete or continuous
  - Time until a projectile returns to earth.
  - The number of times a transistor in computer memory changes state in one operation.
  - The volume of gasoline that is lost to evaporation during the filling of a gas tank.
  - The outside diameter of a machined shaft.

Example of Discrete Random Variable

- Consider toss a fair coin 10 times. The sample space S contains total $2^{10} = 1024$ elements, which is of the form

$$S = \{TTTTTTTTTT, \ldots, HHHHHHHHHH\}$$

- Define the random variable Y as the number of tails out of 10 trials. Remeber that a random variable is a map from sample space to real number. For instance,

$$Y(TTTTTTTTTT) = 10.$$

- The range (all possible values) of Y is $\{0, 1, 2, 3, \ldots, 10\}$, which is much smaller than S.

Example: Mechanical Components

- An assembly consists of three mechanical components. Suppose that the probabilities that the first, second, and third components meet specifications are 0.90, 0.95 and 0.99 respectively. Assume the components are independent.
- Define event
  S_i = the i^{th} component is within specification, where $i = 1, 2, 3$.
- One can calculate
  $P(S_1 S_2 S_3) = (0.9)(0.95)(0.99) = 0.84645$,
  $P(S_1 \overline{S_2}\overline{S_3}) = (0.9)(0.05)(0.01) = 0.00045$, etc.

Example: Mechanical Components

Possible Outcomes for One assembly is

$S_1 S_2 S_3$	$S_1 \overline{S_2}\,\overline{S_3}$	$\overline{S_1} S_2 \overline{S_3}$	$\overline{S_1}\,\overline{S_2} S_3$	$S_1 S_2 \overline{S_3}$	$S_1 \overline{S_2} S_3$	$\overline{S_1} S_2 S_3$	$\overline{S_1}\,\overline{S_2}\,\overline{S_3}$
0.84645	0.00045	0.00095	0.00495	0.00855	0.04455	0.09405	0.00005

Let Y = Number of components within specification in a randomly chosen assembly. Can you fill in the following table?

Y	0	1	2	3
$P(Y = y)$				

Note: We usually denote the realized value of the variable by corresponding lowercase letters.

Probability Mass Functions of Discrete Variables

- Definition: Let X be a discrete random variable defined on some sample space S. The **probability mass function** (PMF) associated with X is defined to be

$$p_X(x) = P(X = x).$$

- A pmf $p(x)$ for a discrete random variable X satisfies the following:
  1. $0 \leq p(x) \leq 1$, for all possible values of x.
  2. The sum of the probabilities, taken over all possible values of X, must equal 1; i.e.,

$$\sum_{\text{all } x} p_X(x) = 1.$$

- Definition: Let Y be a random variable, the **cumulative distribution function** (CDF) of Y is defined as

$$F_Y(y) = P(Y \leq y).$$

- $F_Y(y) = P(Y \leq y)$ is read, "the probability that the random variable Y is less than or equal to the value y."
- Property of cumulative distribution function
    1. $F_Y(y)$ is a nondecreasing function.
    2. $0 \leq F_Y(y) \leq 1$, since $F_Y(y) = P(Y \leq y)$ is a probability.

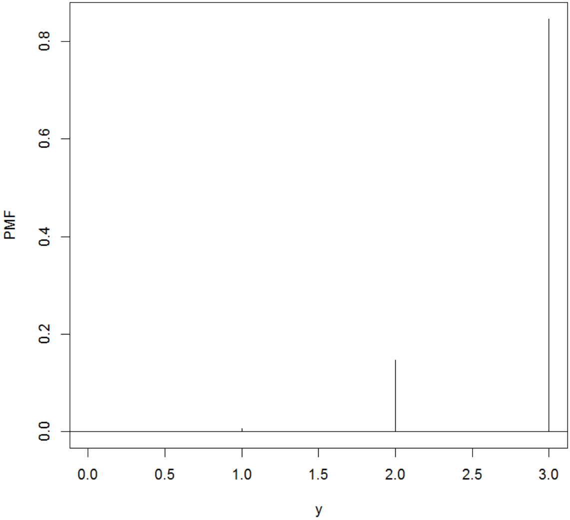
Example: Mechanical Components Revisited

- In the Mechanical Components example Y = Number of components within specification in a randomly chosen assembly. We have the following pmf:

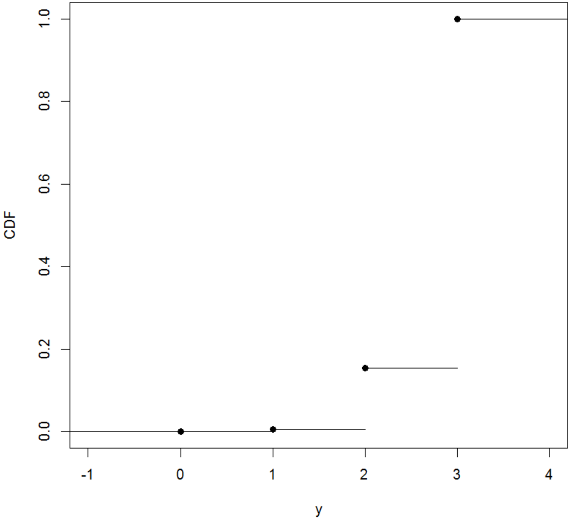
Y	0	1	2	3
$P(Y = y)$	0.00005	0.00635	0.14715	0.84645

- And the CDF is?

Example: PMF plot of Mechanical Components



Example: CDF plot of Mechanical Components



Simulation R Code

```
## PMF plot code
y <- c(0,1,2,3)
pmf <- c(0.00005, 0.00635, 0.14715, 0.84645)
plot(y, pmf, type="h", xlab="y", ylab="PMF")
abline(h=0)

## CDF plot code
cdf <- c(0, cumsum(pmf))
cdf.plot <- stepfun(y, cdf, f=0)
plot.stepfun(cdf.plot, xlab="y", ylab="CDF",
  verticals=FALSE, main="", pch=16)
```

Expected Value of a Random Variable

- Definition: Let Y be a discrete random variable with pmf $p_Y(y)$, the **expected value** or **expectation** of Y is defined as

$$\mu = \mathrm{E}(Y) = \sum_{\text{all } y} y p_Y(y).$$

  The expected value for a discrete random variable Y is simply a weighted average of the possible values of Y. Each value y is weighted by its probability $p_Y(y)$.

- In statistical applications, $\mu = \mathrm{E}(Y)$ is commonly called the **population mean**.

Expected Value: Mechanical Components

- The pmf of Y in the mechanical components example is

Y	0	1	2	3
$P(Y=y)$	0.00005	0.00635	0.14715	0.84645

- The expected value of Y is

$$\begin{aligned} \mu &= \mathrm{E}(Y) = \sum_{\text{all } y} y p_Y(y) \\ &= 0(.00005) + 1(.00635) + 2(.14715) + 3(.84645) \\ &= 2.84 \end{aligned}$$

- Interpretation: **On average**, we would expect 2.84 components within specification in a randomly chosen assembly.

Expected Value as the Long-run Average

- The expected value can be interpreted as the long-run average.
- For the mechanical components example, if we randomly choose an assembly, and record the number of components within specification. Over the **long run**, the **average** of these Y observation would be close (converge) to 2.84.

Expected Value of Functions of Y

- Definition: Let Y be a discrete random variable with pmf $p_Y(y)$. Let g be a real-valued function defined on the range of Y. The **expected value** or **expectation** of $g(Y)$ is defined as

$$\mathrm{E}\left[g(Y)\right] = \sum_{\text{all } y} g(y) P_Y(y).$$

- Interpretation: The expectation $\mathrm{E}\left[g(Y)\right]$ could be viewed as the weighted average of the function g when evaluated at the random variable Y.

Properties of Expectation

Let Y be a discrete random variable with pmf $p_Y(y)$. Suppose that $g_1, g_2, \ldots, g_k$ are real-valued function, and let c be a real constant. The expectation of Y satisfies the following properties:

1. $\mathrm{E}[c] = c$.
2. $\mathrm{E}[cY] = c\mathrm{E}[Y]$.
3. Linearity: $\mathrm{E}\left[\sum_{j=1}^{k} g_j(Y)\right] = \sum_{j=1}^{k} \mathrm{E}[g_j(Y)]$.

Remark 1: The 2nd and 3rd rule together imply that

$$\mathrm{E}\left[\sum_{j=1}^{k} c_j g_j(Y)\right] = \sum_{j=1}^{k} c_j \mathrm{E}[g_j(Y)],$$

for constant c_j.

Remark 2: Suppose X and Y are independent, then $E(XY) = E(X)E(Y)$.

Example: Mechanical Components

Question: In the Mechanical Components example, $Y =$ Number of components within specification in a randomly chosen assembly. We found $\mathrm{E}(Y) = 2.84$. Suppose that the cost (in dollars) to repair the assembly is given by the cost function $g(Y) = (3 - Y)^2$. What is the expected cost to repair an assembly?

$$
\begin{aligned}
\mathrm{E}\left[(3 - Y)^2\right] &= \sum_{\text{all } y} (3 - y)^2 p_Y(y) \\
&= (0 - 3)^2(.00005) + (1 - 3)^2(.00635) \\
&\quad + (2 - 3)^2(.14715) + (3 - 3)^2(.84645) \\
&= 9(.00005) + 4(.00635) + 1(.14715) + 0(.84645) \\
&= 0.173(\text{dollars})
\end{aligned}
$$

Interpretation: One average, the repair cost on each assembly is \$0.173.

Example: Project Management

A project manager for an engineering firm submitted bids on three projects. The following table summarizes the firms chances of having the three projects accepted.

Project	A	B	C
Prob of accept	0.30	0.80	0.10

Question: Assuming the projects are independent of one another, what is the probability that the firm will have all three projects accepted?
Solution:

Example: Project Management Cont'd

Question: What is the probability of having at least one project accepted?

Project	A	B	C
Prob of accept	0.30	0.80	0.10

Solution:

Example: Project Management Cont'd

Question: Let Y = number of projects accepted. Fill in the following table and calculate the expected number of projects accepted.

Y	0	1	2	3
$p_Y(y)$	0.126			

Solution:

Example: Project Management Cont'd

Question: Suppose the average labor costs for the projects decreases with more projects. The relationship between the number of projects and the average labor costs is: average cost $= \frac{1}{Y^2}$ (million dollars). What is the expected average labor cost?

Y	0	1	2	3
$p_Y(y)$	0.126	0.572	0.278	0.024

Solution:

Variance of a Random Variable

- Definition: Let Y be a discrete random variable with pmf $p_Y(y)$ and expected value $\mathrm{E}(Y) = \mu$. The **variance** of Y is given by

$$\sigma^2 \equiv \mathrm{Var}(Y) \equiv \mathrm{E}\left[(Y-\mu)^2\right] = \sum_{\text{all } y} (y-\mu)^2 p_Y(y).$$

  Warning: Variance is **always** non-negative!

- Definition: The **standard deviation** of Y is the positive square root of the variance:

$$\sigma = \sqrt{\sigma^2} = \sqrt{\mathrm{Var}(Y)}.$$

Remarks on Variance (σ^2)

Suppose Y is a random variable with mean μ and variance σ^2.

- The variance is the average **squared distance** from the mean μ.
- The variance of a random variable Y is a measure of **dispersion** or **scatter** in the possible values for Y.
- The larger (smaller) σ^2 is, the more (less) spread in the popssible values of Y about the population mean $\mathrm{E}(Y)$.
- **Computing Formula:**
  $\mathrm{Var}(Y) = \mathrm{E}[(Y - \mu)^2] = \mathrm{E}[Y^2] - [E(Y)]^2$.

Properties on Variance (σ^2)

Suppose X and Y are random variable with finite variance. Let c be a constant.

- $\mathrm{Var}\,(c) = 0$.
- $\mathrm{Var}\,[cY] = c^2\mathrm{Var}\,[Y]$.
- Suppose X and Y are independent, then

$$\mathrm{Var}\,[X+Y] = \mathrm{Var}\,[X] + \mathrm{Var}\,[Y].$$

Question: $\mathrm{Var}\,[X-Y] = \mathrm{Var}\,[X] - \mathrm{Var}\,[Y]$.

- TRUE
- FALSE.

Bernoulli Trials

Definition: Many experiments can be envisioned as consisting of a sequence of "trials," these trails are called Bernoulli trails if

1. each trial results in a "success" or "failure";
2. the trials are independent;
3. the probability of a "success" in each trial, denoted as p, $0 < p < 1$, is the same on every trial.

Examples: Bernoulli Trials

1. When circuit boards used in the manufacture of Blue Ray players are tested, the long-run percentage of defective boards is 5 percent.
    - trial = test circuit board
    - success = defective board is observed
    - $p = P(\text{"success"}) = P(\text{"defective board"}) = 0.05$
2. Ninety-eight percent of all air traffic radar signals are correctly interpreted the first time they are transmitted.
    - trial = radar signal
    - success = signal is correctly interpreted
    - $p = P(\text{"success"}) = P(\text{"correct interpretation"}) = 0.98$

Expectation and Variance of Bernoulli Trials

- Usually, a success is recorded as "1" and a failure as "0".
- Define X to be a **Bernoulli** random variable.
- The probability to success is equal to the probability to get 1.
- Let $p = P(X = 1)$ and $q = 1 - p = P(X = 0)$.
- We have

$$E(X) = p$$

  and

$$\mathrm{Var}(X) = p(1 - p) = pq$$

Binomial Distribution

- Suppose that n Bernoulli trials are performed. Define

$$Y = \text{the number of successes (out of } n \text{ trials performed).}$$

- The random variable Y has a **binomial distribution** with number of trials n and success probability p.
- Shorthand notation is $Y \sim b(n, p)$.
- Let us derive the pmf of Y through an example.

Example: Water Filters

- A manufacturer of water filters for refrigerators monitors the process for defective filters. Historically, this process averages 5% defective filters. Five filters are randomly selected.
- Find the probability that no filters are defective.
  solution:
- Find the probability that exactly 1 filter is defective.
  solution:
- Find the probability that exactly 2 filter is defective.
  solution:
- Can you see the pattern?

Probability Mass Function of Binomial Random Variable

- Suppose $Y \sim b(n, p)$.
- The pmf of Y is given by

$$p(y) = \begin{cases} \binom{n}{y} p^y (1-p)^{n-y} & \text{for } y = 0, 1, 2, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

- Recall that $\binom{n}{r}$ is the number of ways to choose r distinct **unordered** objects from n distinct objects:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Expectation and Variance of Binomial Random Variable

- If $Y \sim b(n, p)$, then

$$\mathrm{E}(Y) = np$$
$$\mathrm{Var}(Y) = np(1-p).$$

- Proof:

Example: Radon Levels

Question: Historically, 10% of homes in Florida have radon levels higher than that recommended by the EPA. In a random sample of 20 homes, find the probability that exactly 3 have radon levels higher than the EPA recommendation. Assume homes are independent of one another.

Solution: Checking each house is a Bernoulli trial, which satisfies

- ✓ Two outcomes: higher than EPA recommendation or satisfies EPA recommendation.
- ✓ Homes are independent of one another.
- ✓ The case that the radon level is higher than the recommendation is considered as a "success". The success probability remains 10%.

Example: Radon Levels Cont'd

So, the binomial distribution is applicable in this example. Define

$$Y = \text{number of home having radon level higher than EPA.}$$

We have $Y \sim b(20, 0.1)$.

$$P(Y = 3) = \binom{20}{3} 0.1^3 0.9^{20-3} = 0.1901.$$

Doing calculation by `R` is much easier:

```
> dbinom(3,20,0.1)
[1] 0.1901199
```

Example: Radon Levels Cont'd

Q: What is the probability that no more than 5 homes out of the sample having higher radon level than recommended?
A: We want to calculate $P(Y \leq 5)$, which is

$$P(Y \leq 5) = P(Y = 0) + P(Y = 1) + ... + P(Y = 5) = 0.9887$$

It takes time to do the calculation by hand! It is much easier to use R:

```
> pbinom(5,20,0.1)
[1] 0.9887469
```

Example: Radon Levels Cont'd

More questions:

- What is the probability that at least 5 homes out of the sample having higher randon level than recommended?
  Solution:
- what is the probability that 2 to 8 homes out of the sample having higher random level than recommended?
  Solution:
- what is the probability that greater than 2 and less than 8 homes out of the sample having higher random level than recommended?
  Solution:

Geometric Distribution

- The geometric distribution also arises in experiments involving Bernoulli trials:
  1. each trial results in a "success" or "failure";
  2. the trials are independent;
  3. the probability of a "success" in each trial, denoted as p, $0 < p < 1$, is the same on every trial.

- Definition: Suppose that Bernoulli trials are continually observed. Define

$$Y = \text{the number of trials to observe the } \textbf{first} \text{ success.}$$

- We say that Y has a **geometric distribution** with success probability p. **Notation:** $Y \sim \text{geom}(p)$.

PMF of Geometric Distribution

If $Y \sim \text{geom}(p)$, then the probability mass function (pmf) of Y is given by

$$p_Y(y) = \begin{cases} (1-p)^{y-1}p & y = 1, 2, 3, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Example: Geometric Distribution

The probability that a wafer contains a large particle of contamination is 0.01. If it is assumed that the wafers are independent, what is the probability that exactly 125 wafers need to be analysed until a large particle is detected?

Solution: Let Y denote the number of samples analysed until a large particle is detected. Then Y is a geometric random variable with $p = 0.01$. The requested probability is

$$P(Y = 125) = (0.01)(0.99)^{125-1} = 0.0029$$

Suppose $Y \sim \text{geom}(p)$, the expected value of Y is

$$\mathrm{E}\,(Y) = \frac{1}{p}.$$

Proof.

Let $q = 1 - p$, $E(Y) = \sum_{y=1}^{\infty} yq^{y-1}p = p\sum_{y=1}^{\infty} yq^{y-1}$. The right-hand side is the derivative of $p\sum_{y=1}^{\infty} q^y$ w.r.t. q. By geometric series,

$$p\sum_{y=1}^{\infty} q^y = \frac{pq}{1-q}.$$

It follows that $\mathrm{E}\,(Y) = \frac{\partial}{\partial q}\left[\frac{pq}{1-q}\right] = \frac{1}{p}$ $\square$

Mean and Variance of Geometric Distribution

The variance can be derived in a similar way, which is given by

$$\mathrm{Var}\left(Y\right) = \frac{1 - p}{p^2}$$

Example Revisit

The probability that a wafer contains a large particle of contamination is 0.01. If it is assumed that the wafers are independent, then on average, how many wafers need to be analysed until a large particle is detected?

Solution:

Example: Fruit Fly

Biology students are checking the eye color of fruit flies. For each fly, the probability of observing white eyes is $p = 0.25$. We interpret

- trail = fruit fly
- success = fly has white eyes
- $p = P(\text{white eyets}) = 0.25$

If the Bernoulli trial assumptions hold (independent and idential), then

$$Y = \text{the number of flies needed to find the first white-eyed}$$
$$\sim geom(p = 0.25)$$

Eample: Fruit Fly

1. What is the probability the first white-eyed fly is observed on the fifth fly checked?

$$P(Y = 5) = (0.25)(1 - 0.25)^{5-1} = (0.25)(1 - 0.25)^4 = 0.079$$

2. What is the probability the first white-eyed fly is observed before the fourth fly is examined?

$$\begin{aligned} &P(Y \leq 3) \\ &= (0.25)(1 - 0.25)^{1-1} + (0.25)(1 - 0.25)^{2-1} + (0.25)(1 - 0.25)^{3-1} \\ &= 0.25 + 0.1875 + 0.140625 \\ &= 0.578 \end{aligned}$$

Geometric Distribution PMF CDF R Code

We can use R to calculate the pmf and cdf of geometric distribution directly by using

$$p_Y(y) = P(Y = y) = \text{dgeom(y-1, p)}$$

and

$$F_Y(y) = P(Y \leq y) = \text{pgeom(y-1, p)}$$

Negative Binomial Distribution

- The negative binomial distribution also arises in experiments involving Bernoulli trials:
  1. each trial results in a "success" or "failure";
  2. the trials are independent;
  3. the probability of a "success" in each trial, denoted as p, $0 < p < 1$, is the same on every trial.

- Definition: Suppose that Bernoulli trials are continually observed. Define

$$Y = \text{the number of trials to observe the } r\textbf{th} \text{ success.}$$

- We say that Y has a **negative binomial distribution** with waiting parameter r and success probability p. **Notation:** $Y \sim \text{nib}(r, p)$.

Negative Binomial Distribution

- The negative binomial distribution is a generalization of the geometric distribution. If $r = 1$, then $\text{nib}(1, p) = \text{geom}(p)$
- If $Y \sim \text{nib}(r, p)$, then the probability mass function of Y is given by

$$p_Y(y) = \begin{cases} \binom{y-1}{r-1} p^r (1-p)^{y-r} & y = 1, 2, 3, \dots \\ 0 & \text{otherwise.} \end{cases}$$

- Mean and Variance:

$$E(Y) = \frac{r}{p}$$
$$\text{var}(Y) = \frac{r(1-p)}{p^2}$$

Example: Fruit Fly Revisit

Biology students are checking the eye color of fruit flies. For each fly, the probability of observing white eyes is $p = 0.25$. What is the probability the third white-eyed fly is observed on the tenth fly checked?

Solution: $Y \sim \text{nib}(r = 3, p = 0.25)$. Therefore,

$$P(Y = 10) = \binom{10-1}{3-1} 0.25^3 (1 - 0.25)^{10-3} = 0.075$$

Negative Binomial Distribution PMF CDF R Code

We can use R to calculate the pmf and cdf of negative binomial distribution directly by using

$$p_Y(y) = P(Y = y) = \text{dnbinom(y-r, r, p)}$$

and

$$F_Y(y) = P(Y \leq y) = \text{pnbinom(y-r, r, p)}$$

Hypergeometric Distribution

Setting: A set of N objects contains r objects classified as successes $N - r$ objects classified as failures. A sample of size n objects is selected randomly (**without replacement**) from the N objects. Define

$$Y = \text{the number of success (out of the } n \text{ selected).}$$

We say that Y has a **hypergeometric distribution** and write $Y \sim \text{hyper}(N, n, r)$. The pmf of Y is given by

$$p(y) = \begin{cases} \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}, & y \leq r \text{ and } n - y \leq N - r \\ 0, & \text{otherwise.} \end{cases}$$

Mean and Variance of Hypergeometric Distribution

If $Y \sim \text{hyper}(N, n, r)$, then

- $E(Y) = n(\frac{r}{N})$
- $\mathrm{Var}\,(Y) = n(\frac{r}{N})(\frac{N-r}{N})(\frac{N-n}{N-1})$.

Example: Hypergeometric Distribution

A batch of parts contains 100 parts from a local supplier of tubing and 200 parts from a supplier of tubing in the next state. If four parts are selected randomly and without replacement, what is the probability they are all from the local supplier?

Solution: Let us identify the parameters first. In this example, $N =$, $r =$, and $n =$.

Let Y equal the number of parts in the sample from the local supplier. Then,

$$P(Y = 4) = \frac{\binom{100}{4}\binom{200}{0}}{\binom{300}{4}} = 0.0119$$

Example: Hypergeometric Distribution

The R code to calculate $P(Y = y)$ is of the form dhyper(y,r,N-r,n). Using R,

```
> dhyper(4,100,200,4)
[1] 0.01185408
```

What is the probability that two or more parts in the sample are from the local supplier?

$$P(Y \geq 2) = 1 - P(Y \leq 1) = 1 - 0.5925943,$$

where $P(Y \leq 1)$ can be computed via R:

```
> phyper(1,100,200,4)
[1] 0.5925943
```

It's your turn...

If a shipment of 100 generators contains 5 faulty generators, what is the probability that we can select 10 generators from the shipment and not get a faulty one?

(a) $\frac{\binom{5}{1}\binom{95}{9}}{\binom{100}{10}}$

(b) $\frac{\binom{5}{0}\binom{95}{9}}{\binom{100}{9}}$

(c) $\frac{\binom{5}{0}\binom{95}{10}}{\binom{100}{10}}$

(d) $\frac{\binom{5}{10}\binom{95}{0}}{\binom{100}{10}}$

- Consider a process that produces insulated copper wire. Historically the process has averaged 2.6 breaks in the insulation per 1000 meters of wire. We want to find the probability that 1000 meters of wire will have 1 or fewer breaks in insulation?
- Is this a binomial problem?
- Is this a hypergeometric problem?

Poisson Distribution

Note: The Poisson distribution is commonly used to model counts, such as

1. the number of customers entering a post office in a given hour
2. the number of α-particles discharged from a radioactive substance in one second
3. the number of machine breakdowns per month
4. the number of insurance claims received per day
5. the number of defects on a piece of raw material.

Poisson Distribution Cont'd

- Poisson distribution can be used to model the number of events occurring in a continuous time or space.
- Let λ be the **average number** of occurrences per **base unit** and t is the number of base units inspected.
- Let Y = the number of "occurrences" over in a unit interval of time (or space). Suppose Poisson distribution is adequate to describe Y. Then, the pmf of Y is given by

$$p_Y(y) = \begin{cases} \frac{(\lambda t)^y e^{-\lambda t}}{y!}, & y = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

- The shorthand notation is $Y \sim \text{Poisson}(\lambda t)$.

Mean and Variance of Poisson Distribution

If $Y \sim \text{Poisson}(\lambda t)$,

$$\mathrm{E}(Y) = \lambda t$$
$$\mathrm{Var}(Y) = \lambda t.$$

Go Back to Insulated Wire

- "Historically the process has averaged 2.6 breaks in the insulation per 1000 meters of wire" implies that the average number of occurrences $\lambda = 2.6$, and the base units is 1000 meters. Let $Y=$ number of breaks 1000 meters of wire will have. So $Y \sim$ Poisson(2.6).
- We have

$$P(Y = 0 \cup Y = 1) = \frac{2.6^0 e^{-2.6}}{0!} + \frac{2.6^1 e^{-2.6}}{1!}$$

- Using R,

```
> dpois(0,2.6)+dpois(1,2.6)
[1] 0.2673849
> ppois(1,2.6)
[1] 0.2673849
```

Go Back to Insulated Wire Cont'd

- If we were inspecting 2000 meters of wire, $\lambda t = (2.6)(2) = 5.2$.

$$Y \sim \text{Poisson}(5.2).$$

- If we were inspecting 500 meters of wire, $\lambda t = (2.6)(0.5) = 1.3$.

$$Y \sim \text{Poisson}(1.3)$$

.

Conditions for a Poisson Distribution

- Areas of inspection are independent of one another.
- The probability of the event occurring at any particular point in space or time is negligible.
- The mean remains constant over all areas of inspection.

The pmf of Poisson distribution is derived based on these three assumption! Check out the derivation by yourself at: `http://www.pp.rhul.ac.uk/~cowan/stat/notes/PoissonNote.pdf`

Questions

1. Suppose we average 5 radioactive particles passing a counter in 1 millisecond. What is the probability that exactly 3 particles will pass in one millisecond?
   Solution:

2. Suppose we average 5 radioactive particles passing a counter in 1 millisecond. What is the probability that exactly 10 particles will pass in three milliseconds?
   Solution: